We can calculate the perimeter and the area of an inscribed regular polygon, if we know only the number of sides and the radius of the circle in which the polygon is inscribed. First we need to find the length of one side of the polygon.

**EXAMPLE 1**

**Find the sidelength of an inscribed regular polygon**

Derive a formula for the length of one side of a regular polygon with \( n \) sides inscribed in a circle of radius \( r \).

**Solution:**

The diagram at the right shows part of a regular polygon, with \( n \) sides, one of which is \( AB \), inscribed in a circle of radius \( r \). A perpendicular, \( AN \), is drawn to side \( AB \). Since the sum of all the angles at vertex \( O \) formed by successive radii add up to 360°, we can see that one of these angles, like \( \angle AOB \), has measure \( \frac{360}{n} \). Therefore, an angle such as \( \angle AON \) has measure

\[
\frac{1}{2} \cdot \frac{360}{n} = \frac{180}{n}.
\]

Also, since \( \sin \angle AON = \frac{AN}{AO} = \frac{AN}{r} \), we know that:

\[
AN = r \sin \frac{180}{n}.
\]

Finally, knowing \( AN \), we can find the length of one side \( s \) of the polygon, because we know that the altitude of an isosceles triangle bisects the base of the triangle. Thus,

\[
s = 2AN = 2r \sin \frac{180}{n}.
\]

**EXAMPLE 2**

**Find the perimeter of an inscribed regular polygon**

Derive a formula for the perimeter of a regular polygon with \( n \) sides inscribed in a circle of radius \( r \).

**Solution:**

The perimeter \( p \) of a polygon with \( n \) sides is given by the formula:

\[
p = ns = n \cdot 2r \sin \frac{180}{n}, \quad \text{or} \quad p = 2nr \sin \frac{180}{n}.
\]

You should be able to see geometrically what number \( p \) gets close to when \( n \) increases, by simply imagining a regular polygon with many sides inscribed in a circle. In the Practice, you will verify your geometric intuition algebraically.

**EXAMPLE 3**

**Find the area of an inscribed regular polygon**

Derive a formula for the area of a regular polygon with \( n \) sides inscribed in a circle of radius \( r \).

**Solution:**

As the diagram above indicates, the area \( A \) of a regular polygon is given by the formula

\[
A = nK
\]
where $K$ is the area of one of the $n$ congruent isosceles triangles (like $\triangle AOB$), that are formed by consecutive radii of the inscribed polygon (like $OA$) drawn to the vertices, and with one side of the polygon as the base. In order to find $K$ in terms of $n$ and $r$, we need to express $ON$ in terms of these quantities. Since $\cos \angle AON = \frac{ON}{AO} = \frac{ON}{r}$, we know that:

$$ON = r \cos \frac{180}{n}.$$  

Using the formula $\text{Area} = \left( \frac{1}{2} \right) \text{(height)} \cdot \text{(base)}$ for the area of a triangle, we have:

$$K = \frac{1}{2} (ON)(AB) = \frac{1}{2} r \cos \frac{180}{n} \cdot 2 r \sin \frac{180}{n}.$$  

This can be simplified to:

$$K = r^2 \cos \frac{180}{n} \cdot \sin \frac{180}{n}.$$  

Therefore, since $A = nK$, an area formula for the whole polygon is given by:

$$A = nr^2 \cos \frac{180}{n} \cdot \sin \frac{180}{n}.$$  

The formula found in the last example can be further simplified, as you will show in the Practice below.

**Practice**

1. Find the perimeter and area of a polygon with $n$ sides inscribed in a circle of radius 5.
   - a. $n = 10$
   - b. $n = 20$
   - c. $n = 70$
   - d. $n = 100$

2. In calculus, you will show that as $n$ gets larger and larger, the number $n \sin \frac{180}{n}$ (where the angle is in degrees, as usual) gets closer and closer to a fixed number. By evaluating this expression for $n = 10, 20, 70, \text{and } 100$ on your calculator, guess the value of this fixed number. What does this tell you about the perimeter of a regular polygon as $n$ gets larger and larger? Explain this fact geometrically.

3. The sine of twice a given angle is *not* equal to twice the sine of the angle, but there is a trigonometric formula that tells you how to find the sine of twice an angle if you know the sine and cosine of the angle itself. Explicitly, the formula says that:

$$\sin 2\theta = 2 \sin \theta \cos \theta.$$  

Use this formula to simplify the formula for the area of a regular polygon derived above.

4. Find the area of the circle in which the polygons in Practice 1–3 are inscribed. Comparing this area to the areas of the polygons themselves, what do you notice?
5. Suppose \( R, S, T, V, W, X, \ldots \) are the vertices of a regular polygon inscribed in a circle of radius \( r \), as shown (the sides of the polygon are not drawn, because they would almost coincide with the part of the circle shown). Suppose also that \( N \) is the midpoint of arc \( TV \).

\( \textbf{a.} \) Express the approximate area of triangle \( TOV \) in terms of \( ON \) and the length \( TV \) of chord \( TV \) (not drawn).

\( \textbf{b.} \) Let \( C \) represent the circumference of the circle. Then you know that \( C = 2\pi r \). Express \( C \) approximately in terms of \( TV \) and \( n \). Equate these two expressions for \( C \) and solve for an approximate value for \( TV \).

\( \textbf{c.} \) The area of the circle is approximately equal to the area of \( n \) triangles congruent to \( \triangle TOV \). Use this fact to express the approximate area of the circle in terms of \( ON \) and \( TV \).

\( \textbf{d.} \) Substitute the expression for \( TV \) from Part(b) into the answer to Part(c) and simplify it, using the fact that \( ON = r \). This gives you a formula for the approximate area of a circle.

\( \textbf{e.} \) Note that the formula for the area of the circle that you found in Part(d) is only an \textit{approximation}, but one that gets better and better as \( n \) gets larger. On the other hand, the expression does not contain \( n \). What does this tell you about the formula?